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A Renormalization Group Explanation of Numerical Observations of Analyticity Domains

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A recent paper considers the dependence of the size of analyticity domains of some functions appearing in KAM theory as a function of the distance to breakdown. They tentatively conclude that the relation is linear. In this note we argue that McKay's renormalization group picture predicts a power-law dependence with an exponent close to 1 but not equal to 1.

KEY WORDS: Analyticity domains; breakup of tori; renormalization group.

1. INTRODUCTION

A recent paper[1] studies the analyticity properties of functions appearing in the KAM theory of twist mappings.

This theory is concerned with families of maps $T_\varepsilon\colon \mathbf{T}^1 \times \mathbf{R} \to \mathbf{T}^1 \times \mathbf{R}$ which preserve area, satisfy a "twist" and "no-flux" condition, and for which T_0 is integrable, i.e., $T_0(\varphi, A) = (\varphi + \omega(A), A)$.

The canonical example to study has been the "standard" family

$$T_\varepsilon(\varphi, A) = \left(\varphi + A + \varepsilon \frac{1}{2\pi} \sin(2\pi\varphi), A + \varepsilon \frac{1}{2\pi} \sin(2\pi\varphi)\right) \tag{1.1}$$

and the goal is to find topologically nontrivial invariant circles which are invariant under the map and on which the motion can be reduced to a rotation.

Analytically, such circles can be conveniently described by finding an embedding of the circle into the torus $K\colon \mathbf{T}^1 \to \mathbf{T}^1 \times \mathbf{R}$ that satisfies

$$T_\varepsilon(K_\varepsilon(\theta)) = K_\varepsilon(\theta + \omega) \tag{1.2}$$

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Hence, the analytic problem may be reduced to finding $K_\varepsilon\colon \mathbf{R}^1 \times \mathbf{R}^1$ satisfying (1.2) and two normalization conditions

$$\begin{aligned} &K_\varepsilon(\theta+1)=K_\varepsilon(\theta) \\ &\int K_\varepsilon(\theta)\, d\theta = 0 \end{aligned} \tag{1.3}$$

The first condition ensures that K_ε indeed is a mapping from the circle to the annulus and the second is a condition that makes the map unique and prevents us from doing something silly, such as profiting from the ambiguity of the previous two requirements. [If $K_\varepsilon(\theta)$ solves (1.2) and the first normalization, so does $K_\varepsilon(\theta + a_\varepsilon) - a_\varepsilon$.]

Rather than using the function $K_\varepsilon(\theta)$, it is customary to use the function $u_\varepsilon(\theta)$ related to $K_\varepsilon(\theta)$ by

$$K_\varepsilon(\theta) = (u_\varepsilon(\theta),\, u_\varepsilon(\theta) - u_\varepsilon(\theta - \omega)) \tag{1.4}$$

The goal of ref. 1 is to study the analyticity properties in ε, θ of $K_\varepsilon(\theta)$—or of $u_\varepsilon(\theta)$.

In this note, we will concern ourselves with the considerations of analyticity in θ. We will use the function K_ε rather than u_ε. From (1.4) it is clear that

$$\mathrm{Dom}_\theta(K_\varepsilon) = \mathrm{Dom}_\theta(u_\varepsilon) \cap (\mathrm{Dom}_\theta(u_\varepsilon) - \omega)$$

From (1.2) it is clear that $\theta_0 \in \mathrm{Dom}_\theta(K_\varepsilon) \Rightarrow \theta_0 + \omega \in \mathrm{Dom}_\theta(K_\varepsilon)$. Hence the domain of K_ε is a strip bounded by horizontal straight lines.

$$\mathrm{Dom}_\theta(K_\varepsilon) = \{\theta \mid |\mathrm{Im}\, \theta| \leqslant \gamma(\varepsilon)\}$$

All evidence (numerical and heuristic) points to u_ε having also a domain which is a strip and which, hence, agrees with that of K_ε.

The main goal of interest will be $\gamma(\varepsilon)$, particularly the behavior of $\gamma(\varepsilon)$ around the point where the circle disappears.

2. RENORMALIZATION THEORY

We recall the "scaling limit" formulation of the renormalization group.

If we parametrize ε in such a way that $\varepsilon = 0$ corresponds to the critical value and choose a coordinate system in such a way that $(0, 0)$ corresponds to a point in the golden invariant curve, we have, for some well-defined numbers δ and matrix A,

$$\frac{1}{A^n} T^{F_n}_{\varepsilon\delta^{-n}} A^n \to T^*_\varepsilon \tag{2.1}$$

where F_n is the nth Fibonacci and T^*_ε is a universal family, which can be described as the unstable manifold of a renormalization operator.

That is, if we perform simultaneously changes of scale in time space and parameter, we converge to a well-defined limit.

We observe that (1.2) implies

$$\frac{1}{A^n} T^{F_n}_{\varepsilon\delta^{-n}} A^n \frac{1}{A^n} K_{\varepsilon\delta^{-n}}(\theta) = \frac{1}{A^n} K_{\varepsilon\delta^{-n}}(\theta + F_n \omega)$$

We also recall that when ω is the golden mean, $F_n \omega \approx (-\omega)^{n-1} + F_{n-1}$. Hence, we see that if we scale θ by ω, we obtain a well-defined limit

$$\frac{1}{A^n} K_{\varepsilon\delta^{-n}}(\omega^n \theta) \to K^*_\varepsilon$$

and

$$T^*_\varepsilon K^*_\varepsilon(\theta) = K^*_\varepsilon(\theta + \omega)$$

This implies that

$$\gamma(\varepsilon\delta^{-n}) \frac{1}{\omega^n}$$

will have a limit—depending on ε.

Hence,

$$\gamma \approx \varepsilon^\alpha \quad \text{with} \quad \alpha = \frac{-\log \omega}{\log \delta}$$

δ was computed in ref. 5, Eq. (10.4), to be 1.6280. Hence, $\alpha = 0.9874$. Which is within the experimental error of the value reported in ref. 1.

We point out that, given the fact that there are increasing indications that the renormalization group has more complicated behaviors than those controlled by McKay's fixed point,[2,3,6,7] it would be quite interesting to compute whether scaling behaviors exist for other families. In doing so, we believe it would be quite advantageous to perform direct calculation of solutions based on a Newton method solution of the functional equations rather than sum the expansions in powers of ε, since they are quite close to the domain of convergence and, hence, the sum is numerically unstable.

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